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Staircase polygons and recurrent lattice walks

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In this paper we derive a direct relationship between the staircase-polygon-generating function Z_d of Guttmann and Prellberg [Phys. Rev. E **47**, R2233 (1993)] and the generating function for recurrent lattice walks P_d for the simple (hyper-) cubic lattice in all dimensions d. A recursion formula is obtained for the Z_d with respect to dimension, which leads to a simplified derivation of Guttmann and Prellberg's result for $d=3$, avoiding the use of the Heun function, and a derivation of their formula for $d=4$ from an integral representation is given in the Appendix.

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INTRODUCTION

In an interesting recent paper [1], Guttmann and Prellberg investigated the generating function for staircase polynomials on a d-dimensional hypercubic lattice ($d=1,2,3,\ldots$). For $d=3,4$, due to the occurrence of the same Heun function as in Joyce's [2] treatment of the lattice Green functions for $d=3$, they suggested that there might be a relationship between the generating function

$$Z_d(x^2)=\sum_{k_1,\ldots,k_d=0}\begin{bmatrix}k_1+\cdots+k_d\\ k_1,\ldots,k_d\end{bmatrix}^2 x^{2(k_1+\cdots+k_d)} \qquad (d|x|<1) \quad (1)$$

for squares of multinomial coefficients and the lattice Green functions

$$P_d(z)=\frac{1}{\pi^d}\int\cdots\int_0^{\pi}\frac{dk_1\cdots dk_d}{1-\frac{z}{d}[\cos(k_1)+\cdots+\cos(k_d)]} \qquad (|z|<1)\ . \quad (2)$$

In this Rapid Communication, we shall show that Z_d and P_d are in fact equivalent to one another through the Abel transform. In addition to some related observations, we present in the Appendix a direct evaluation of Z_4, which avoids the introduction of Heun functions.

INTEGRAL REPRESENTATIONS AND THE ABEL EQUATION

By noting the identities

$$(m!)^2=\int_0^{\infty}t(t/2)^{2m}K_0(t)dt\ , \quad (3)$$

$$\frac{1}{z}=\int_0^{\infty}e^{-sz}ds\ , \quad (4)$$

and the representations for the modified Bessel function

$$I_0(z)=\sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{(n!)^2}=\frac{1}{\pi}\int_0^{\pi}e^{z\cos(\theta)}d\theta\ , \quad (5)$$

we obtain the integral representations

$$Z_d(x^2)=\int_0^{\infty}tK_0(t)I_0^d(xt)dt \quad (6a)$$

and

$$P_d(z)=\int_0^{\infty}e^{-s}I_0^d(sz/d)ds\ . \quad (6b)$$

Therefore, since

$$e^{-s}=\frac{2s}{\pi}\int_0^1\frac{du}{u^2\sqrt{1-u^2}}K_0(s/u)\ , \quad (7)$$

we find the relation

$$P_d(z)=\frac{2}{\pi}\int_0^1\frac{Z_d(u^2z^2/d^2)}{\sqrt{1-u^2}}du\ . \quad (8)$$

For the parameter ranges indicated, all the integrals are absolutely convergent; sums and integrals can therefore be interchanged freely.

Equation (8) is easily found to be equivalent to the Abel integral equation [3]

$$P_d(dz^{1/2})-1=\frac{1}{\pi}\int_0^z\frac{\xi^{-1/2}(Z_d(\xi)-1)}{\sqrt{z-\xi}}d\xi \qquad (z\geq 0)\ , \quad (9)$$

where the 1's have been inserted so that the unknown function $\xi^{-1/2}(Z_d(\xi)-1)$ vanishes as $\xi\to 0+$. The inversion of (9) yields

$$Z_d(x^2)=\frac{d}{dx}\left[x\int_0^1\frac{uP_d(dxu)}{\sqrt{1-u^2}}du\right]\ . \quad (10)$$

Equations (8) and (10) prove that Z_d and P_d are simply different representations for the same mathematical object.

As a simple example, since [1] $Z_2(x^2)=(1-4x^2)^{-1/2}$, (8) gives

$$P_2(z)=\frac{2}{\pi}\int_0^1\frac{du}{\sqrt{(1-u^2)(1-z^2u^2)}}=\frac{2}{\pi}K(z)\ , \tag{11}$$

while, since

$$\int_0^1\frac{uK(au)}{\sqrt{1-u^2}}du=\frac{\pi}{2a}\sin^{-1}a\ , \tag{12}$$

(10) is also trivially satisfied. The application of this procedure to the known results for Z_3 and P_3 produces two rather fascinating integrations over the product of two complete elliptic integrals.

Because of the equivalence of Z_d and P_d, new representations for the lattice Green functions are obtained in terms of the combinatorial quantities $S_n^{(d)}$ investigated by Guttmann and Prellberg in [1]. Thus

$$P_d(z)=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!d^{2n}}S_n^{(d)}z^{2n} \tag{13}$$

with

$$S_n^{(d)}=\sum_{m_1=0}^{n}\sum_{m_2=0}^{m_1}\cdots\sum_{m_{d-1}}^{m_{d-2}}\begin{bmatrix}n\\m_1\end{bmatrix}^2\cdots\begin{bmatrix}m_{d-3}\\m_{d-2}\end{bmatrix}^2\begin{bmatrix}m_{d-2}\\m_{d-1}\end{bmatrix}^2 . \tag{14}$$

It is also interesting to observe that

$$Z_{d+1}(x^2)=\frac{1}{\pi(1-x^2)}\int_0^{\pi}Z_d\left[\frac{x^2}{(1-x^2)^2}(1+2x\cos\theta+x^2)\right]d\theta\ . \tag{15}$$

This follows at once from the formulas $I_0^d(z)=\sum_n[(z/2)^{2n}/(n!)^2]S_n^{(d)}$, Eq. (6a), which gives

$$Z_{d+1}(x^2)=\sum_{n=0}^{\infty}\frac{(x/2)^{2n}}{(n!)^2}S_n^{(d)}\int_0^{\infty}t^{2n+1}K_0(t)I_0(xt)dt\ , \tag{16}$$

and, in terms of Gauss's function,

$$\begin{aligned}\int_0^{\infty}t^{2n+1}K_0(t)I_0(xt)dt&=4^n(n!)^2\,{}_2F_1(n+1,n+1;1;x^2)\\&=4^n(n!)^2(1-x^2)^{-(2n+1)}\frac{1}{\pi}\\&\quad\times\int_0^{\pi}(1+2x\cos\theta+x^2)^nd\theta\ .\end{aligned} \tag{17}$$

For instance, since $Z_2(x^2)=(1-4x^2)^{-1/2}$, we have

$$\begin{aligned}Z_3(x^2)&=\frac{1}{\pi}\int_0^{\pi}(1-6x^2-3x^4-8x^3\cos\theta)^{-1/2}d\theta\\&=(1+x)^{-3/2}(1-3x)^{-1/2}\\&\quad\times{}_2F_1\left[\tfrac{1}{2},\tfrac{1}{2};1;-\frac{16x^3}{(1+x)^3(1-3x)}\right]\end{aligned} \tag{18}$$

to be compared with [1] $Z_3(x^2)=F(\frac{1}{9},-\frac{1}{3};1,1,1,1;x^2)$.

Furthermore, the Fuchsian differential equations given for $Z_d(x)$ with $d=2,3,4,5,6$ might be useful for writing down similar equations for the lattice Green functions in these dimensions.

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APPENDIX

Here we include an alternative evaluation of Z_4, which avoids the Heun function.

For $d=4$, (6a) gives

$$Z_4(x^2)=\int_0^{\infty}tK_0(t)I_0^4(xt)dt\ . \tag{A1}$$

By using Watson's series

$$I_0^2(x)=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{(n!)^3}x^{2n}\ , \tag{A2}$$

and the formula

$$\int_0^{\infty}t^{2m+1}K_0(t)dt=4^m(m!)^2\ , \tag{A3}$$

(A1) becomes

$$Z_4(x^2)=\sum_{m,n=0}^{\infty}(4x^2)^{m+n}\frac{(\frac{1}{2})_m(\frac{1}{2})_n}{(m!n!)^3}[(m+n)!]^2\ . \tag{A4}$$

This expression is easily rearranged into

$$Z_4(x^2)=\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{k!}(4x^2)^k\,{}_4F_3\left[\begin{matrix}-k,-k,-k,\frac{1}{2};\\1,1,\frac{1}{2}-k;\end{matrix}\ 1\right]. \tag{A5}$$

Also [4]

$${}_4F_3\left[\begin{matrix}-k,-k,-k,\frac{1}{2};\\1,1,\frac{1}{2}-k;\end{matrix}\ 1\right]=4^k\frac{(\frac{1}{2})_k}{k!}\,{}_4F_3\left[\begin{matrix}-k/2,(1-k)/2,-k,\frac{1}{2};\\1,\frac{1}{2}-k,\frac{1}{2}-k;\end{matrix}\ 1\right], \tag{A6}$$

so by reexpanding the hypergeometric function,

$$Z_4(x^2)=\sum_{n=0}^{\infty}(4x^2)^n\sum_{k=0}^{[n/2]}\left[\frac{(\frac{1}{2})_n}{(\frac{1}{2}-n)_k}\right]^2 4^{n-k}\frac{(-n)_k(\frac{1}{2})_k(-n/2)_k[(1-n)/2]_k4^k}{(n!)^2(k!)^2}\ . \tag{A7}$$

Again, by applying the identity

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} F(n,k)=\sum_{n,m=0}^{\infty} F(n+2k,k)\ , \tag{A8}$$

we obtain

$$\begin{aligned} Z_4(x^2) &= \sum_{m,n=0}^{\infty} \frac{(\frac{1}{2})_{n+m}(\frac{1}{2})_{m+n}(\frac{1}{2})_m}{(n+m)!n!(m!)^2}(-1)^m(4x^2)^{n+2m}4^{n+m} \\ &= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!}(16x^2)^n \sum_{k=0}^{n} \frac{(-1)^k(\frac{1}{2})_k}{(n-k)!(k!)^2}(4x^2)^k \\ &= \sum_{n=0}^{\infty} \left[\frac{(\frac{1}{2})_n}{n!}\right]^2 {}_2F_1(-n,\tfrac{1}{2};1;4x^2)(16x^2)^n\ . \end{aligned} \tag{A9}$$

Now, by using standard hypergeometric identities, this can be reexpressed as an Appell hypergeometric function of two variables [5]:

$$\begin{aligned} Z_4(x^2) &= \sum_{n=0}^{\infty} \left[\frac{(\frac{1}{2})_n}{n!}\right]^2 [4x^2(1+\sqrt{1-4x^2})^2]^n\, {}_2F_1\left[-n,-n;1;\left[\frac{1-\sqrt{1-4x^2}}{1+\sqrt{1-4x^2}}\right]^2\right] \\ &= F_4(\tfrac{1}{2},\tfrac{1}{2};1,1;4x^2(1-\sqrt{1-4x^2})^2,4x^2(1+\sqrt{1-4x^2})^2)\ . \end{aligned} \tag{A10}$$

Finally, by writing

$$\begin{aligned} k_+^2(1-k_-^2) &= 4x^2(1-\sqrt{1-4x^2})^2\ , \\ k_-^2(1-k_+^2) &= 4x^2(1+\sqrt{1-4x^2})^2\ , \end{aligned} \tag{A11}$$

so

$$k_\pm^2=\tfrac{1}{2}\pm 8x^2\sqrt{1-4x^2}-\tfrac{1}{2}(1-8x^2)\sqrt{1-16x^2}\ , \tag{A12}$$

and applying Bailey's theorem, we arrive at Guttmann and Prellberg's formula

$$\begin{aligned} Z_4(x^2) &= {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;k_+^2)\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;k_-^2) \\ &= \frac{4}{\pi^2}K(k_+)K(k_-)\ . \end{aligned} \tag{A13}$$

The key point of this evaluation is (A6). We mention here a further ${}_4F_3$ transformation

$${}_4F_3\left[\begin{matrix} -k,\frac{1}{2},\frac{1}{2},\frac{1}{2}; \\ \frac{1}{2}-k,\frac{1}{2}-k,\frac{1}{2}-k; \end{matrix}\ 1\right] = \frac{k!}{(\frac{1}{2})_k}\,{}_4F_3\left[\begin{matrix} -k,-k/2,(1-k)/2,\frac{1}{2}; \\ \frac{1}{2}-k,\frac{1}{2}-k,1; \end{matrix}\ 1\right] \tag{A14}$$

[[4], p.65, Eq. (2.4.2.3)], which, together with (A6), gives

$${}_4F_3\left[\begin{matrix} -k,-k,-k,\frac{1}{2}; \\ 1,1,\frac{1}{2}-k; \end{matrix}\ 1\right] = 4^k\left[\frac{(\frac{1}{2})_k}{k!}\right]^2 {}_4F_3\left[\begin{matrix} -k,\frac{1}{2},\frac{1}{2},\frac{1}{2}; \\ \frac{1}{2}-k,\frac{1}{2}-k,\frac{1}{2}-k; \end{matrix}\ 1\right]\ . \tag{A15}$$

As an application, consider the fcc lattice Green function

$$I(t)=\frac{1}{\pi^3}\int\int\int_0^{\pi}\frac{1}{1-t(\cos x\,\cos y+\cos x\,\cos z+\cos y\,\cos z)}dx\,dy\,dz\ . \tag{A16}$$

The integration over z gives

$$\frac{\pi}{1+t}(1-4u\,\cos^2 x/2\,\cos^2 y/2)^{-1/2}(1-4u\,\sin^2 x/2\,\sin^2 y/2)^{-1/2}\ , \tag{A17}$$

where $u=t/(1+t)$. Therefore, we have

$$\begin{aligned} \frac{\pi^2}{4}(1+t)I(t) &= \int_0^{\pi/2}\int dx\,dy(1-4u\,\cos^2 x\,\cos^2 y)^{-1/2}(1-4u\,\sin^2 x\,\sin^2 y)^{-1/2} \\ &= \frac{\pi^2}{4}\sum_{r,s=0}^{\infty}\frac{(\frac{1}{2})_r(\frac{1}{2})_s}{r!s!}(4u)^{4+s}\left[\frac{(\frac{1}{2})_r(\frac{1}{2})_s}{(r+s)!}\right]^2 \\ &= \frac{\pi^2}{4}\sum_{k=0}^{\infty}\left[\frac{(\frac{1}{2})_k}{k!}\right]^3(4u)^k\,{}_4F_3\left[\begin{matrix} -k,\frac{1}{2},\frac{1}{2},\frac{1}{2}; \\ \frac{1}{2}-k,\frac{1}{2}-k,\frac{1}{2}-k; \end{matrix}\ 1\right]\ , \end{aligned} \tag{A18}$$

that is, recalling (A5) and (A15),

$$I(t)=\frac{1}{1+t}Z_4(u/4)=\frac{1}{1+t}\frac{4}{\pi^2}K(k_+)K(k_-)\ , \tag{A19}$$

where

$$k_\pm^2=\tfrac{1}{2}\pm 2t(1+t)^{-3/2}-\tfrac{1}{2}(1-t)(1-3t)^{1/2}(1+t)^{-3/2} \tag{A20}$$

[[2] Eq. (7.3)]. Thus, within the framework of the generalized hypergeometric series, the fcc lattice Green function appears to be intimately related to Z_4.

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